ON THE STABILITY OF PERIODIC 2D EULER-α FLOWS

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ABSTRACT. Sectional curvature of the group $\mathcal{D}_\mu(M)$ of volume-preserving diffeomorphisms of a two-torus with the H^1 metric is analyzed. An explicit expression is obtained for the sectional curvature in the plane spanned by two stationary flows, $\cos(k,x)$ and $\cos(l,x)$. It is shown that for certain values of the wave vectors k and l the curvature becomes positive for $\alpha > \alpha_0$, where $0 < \alpha_0 < 1$ is of the order $1/k$. This suggests that the flow corresponding to such geodesics becomes more stable as one goes from usual Eulerian description to the Euler-α model.

CONTENTS

1. INTRODUCTION

In Lagrangian mechanics a motion of a natural mechanical system is a geodesic line on a manifold - configuration space in the metric given by the difference of kinetic and potential energy. The configuration space for the fluid motion in a domain M is the group $\mathcal{D}_\mu(M)$ of volume-preserving diffeomorphisms of M. The corresponding (Lie) algebra is the algebra of divergence-free vector fields on M vanishing on the boundary. The standard (Euler) model of an ideal fluid corresponds to the kinetic energy being given by the L^2 norm of the fluid velocity on M. That is, the right-invariant metric on $\mathcal{D}_\mu(M)$ is defined in the following way: its value at the identity of the group on a divergence-free vector field v from the algebra is given by $\langle v, v\rangle = \|v\|_{l^2} = \int_M (v, v)dx$.

Recently, a number of papers (see, e.g., [HMR, S 98, S 99]) introduced the so called averaged Euler equations for ideal incompressible flow on a manifold M. The averaged Euler equations involve a parameter α; one interpretation is that they are obtained by temporally averaging the Euler equations in Lagrangian representation over rapid fluctuations whose amplitudes are of order α. The particle flows associated with these equations can be shown to be geodesics on a suitable group of volume-preserving diffeomorphisms but with respect to a right invariant H^1 metric instead of the L^2 metric.

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The case of area-preserving diffeomorphisms of the two-dimensional torus with a right invariant L^2 metric was analyzed by Arnold who showed (see, e.g. [A 66, AK 98]) that "in many directions the sectional curvature is negative". In this paper we consider geodesic stability problem for the group $\mathcal{D}_\mu(T^2)$ with a right invariant H^1 metric which is related to the average Euler flows.

The instability discussed in this paper is the exponential *Lagrangian* instability of the motion of the fluid, not of its velocity field. A stationary flow can be a Lyapunov stable solution of Euler equations, while the corresponding motion of the fluid is exponentially unstable. The reason is that a small perturbation of the fluid velocity field can induce exponential divergence of fluid particles.

2. Instability of the Euler flow on T^2

Here we review Arnold's results for the group $\mathcal{D}_\mu(T^2)$ with a right invariant L^2 metric closely following [AK 98]. Recall some standard notations. Let B denote the bilinear form on a Lie algebra $\mathfrak{g}$ defined by the relation $\langle B(\xi,\eta),\zeta\rangle = \langle \xi, [\eta,\zeta]\rangle$, where $\xi,\eta,\zeta \in \mathfrak{g}$ $[\cdot,\cdot]$ is the commutator in $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle$ is the inner product in the space $\mathfrak{g}$.

The (Riemannian) *curvature tensor* R describes the infinitesimal transformation on a tangent space obtained by parallel translation around an infinitely small parallelogram. For $u, v, w \in T_{x_0}M$, the action of $R(u, v)$ on w can be expressed in terms of covariant differentiation as follows

$$R(u,v)w = (-\nabla_{\bar u}\nabla_{\bar v}\bar w + \nabla_{\bar u}\nabla_{\bar v}\bar w + +\nabla_{\{\bar u,\bar u\}}\bar w)|_{x=x_0}, \tag{2.1}$$

where $\bar u, \bar v, \bar w$ are any fields whose values at the point x_0 are u, v, w.

The *sectional curvature* of M in the direction of the two-plane spanned by any two vectors $u, v \in T_{x_0}M$ is the value

$$C_{uv} = \frac{\langle R(u,v)u, v\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}. \tag{2.2}$$

Theorem 3.2 of [AK 98] gives explicit formulas for the inner product, commutator, operation B, connection, and curvature of the right invariant L^2 metric on the group $\mathcal{D}_\mu(T^2)$. These formulas allow one to calculate the sectional curvature in any two-dimensional direction.

The divergence-free vector fields that constitute the Lie algebra of the group $\mathcal{D}_\mu(T^2)$ can be described by their stream (Hamiltonian) functions with zero mean (i.e., $v = -\dfrac{\partial H}{\partial y}\dfrac{\partial}{\partial x} + \dfrac{\partial H}{\partial x}\dfrac{\partial}{\partial y}$). Thus, the Lie algebra can be identified with the space of real functions on the torus having zero average value [AK 98]. It is convenient to define such functions by their Fourier coefficients and to carry out all calculations over $\mathbb{C}$.

Complexifying the Lie algebra one constructs a basis of this vector space using the functions e_k (where k, called a *wave vector*, is a point of $\mathbb{R}^2$) whose value at a point x of our complex plane is equal to $e^{\imath(k,x)}$. This determines a function on the torus if the inner product (k, x) is a multiple of 2π for all $x \in \Gamma$. All such vectors k belong to a lattice Γ^* in $\mathbb{R}^2$, and the functions $\{e_k | k \in \Gamma^*, k \neq 0\}$ form a basis of the complexified Lie algebra.

Consider the parallel sinusoidal steady flow given by the stream function $\xi = \cos(k, x)$ and let η be any other vector of the algebra, i.e. $\eta = \sum x_l e_l$, where $x_{-l} = \bar x_l$. Theorem 3.4 of [AK 98] states that the curvature of the group $\mathcal{D}_\mu(T^2)$ in

any two-dimensional plane containing the direction ξ is *non-positive* and is given by

$$C_{\xi\eta} = \frac{S}{4} \sum_l a_{kl}^2 |x_l + x_{l+2k}|^2, \tag{2.3}$$

where $a_{kl} = \dfrac{(k \times l)^2}{|k+l|}$, $k \times l = k_1 l_2 - k_2 l_1$ is the (oriented) area of the parallelogram spanned by k and l, and S is the area of the torus. Then, a corollary of this theorem states that the curvature in the plane defined by the stream functions $\xi = \cos(k, x)$ and $\eta = \cos(l, x)$ is

$$C_{\xi\eta} = -(k^2 + l^2) \sin^2 \beta \sin^2 \gamma / 4S, \tag{2.4}$$

where β is the angle between k and l, and γ is the angle between $k + l$ and $k - l$.

3. Stable directions for the Euler-α flow on T^2

In this section we present new results on the sectional curvature of the group of area-preserving diffeomorphisms of a two-torus with a right invariant H^1 metric in view of the application to the Lagrangian stability analysis following Arnold [A 66]. The foundations for these results were established in [S 98] where the continuous differentiability of the geodesic spray of H^1 metric on $\mathcal{D}^s_\mu(M)$ for an arbitrary Riemannian manifold M was proved.

We start with an analog of Theorem 3.2 of [AK 98]. Define an operator A^α : $\mathbb{R}^2 \to \mathbb{R}_+$, $\quad k \mapsto k^2(1+\alpha^2 k^2)$. It corresponds to the H^1 norm in the Fourier space and is simply given by k^2 in the case $\alpha = 0$ when the H^1 metric effectively becomes the L^2 metric.

Theorem 3.1. *The explicit formulas for the inner product, commutator, operation B, and connection of the right invariant H^1 metric on the group $\mathcal{D}_\mu(T^2)$ have the following form:*

$$\langle e_k, e_l \rangle = A^\alpha(k) \delta_{k,-l} \tag{3.1}$$

$$[e_k, e_l] = (k \times l) e_{k+l} \tag{3.2}$$

$$B(e_k, e_l) = b_{k,l} e_{k+l}, \quad \text{where} \quad b_{k,l} = (k \times l) \frac{A^\alpha(k)}{A^\alpha(k+l)} \tag{3.3}$$

$$\nabla_{e_k} e_l = d_{k,k+l} e_{k+l}, \quad \text{where} \quad d_{k,k+l} = \frac{k \times l}{s} \left(1 - \frac{A^\alpha(k) - A^\alpha(l)}{A^\alpha(k+l)} \right). \tag{3.4}$$

Using the definition of the curvature tensor (2.1) we obtain

$$\begin{aligned} R_{k,l,m,n} \equiv \langle R(e_k, e_l) e_m, e_n \rangle = (&-d_{l+m,k+l+m} d_{m,l+m} \\ &+ d_{k+m,k+l+m} d_{m,k+m} + (k \times l) d_{m,k+l+m}) A^\alpha(k+l+m) S. \end{aligned} \tag{3.5}$$

We do not write here the explicit expression for $R_{k,l,m,n}$ as it is rather involved, but we note that it is non-zero only in the case $k+l+m+n = 0$. We analyze a special case of the curvature in the plane defined by the stream functions $\xi = \cos(k, x)$ and $\eta = \cos(l, x)$ (notice that the corresponding flow is a solution of the averaged Euler

equations). Then the sectional curvature is determined only by two terms (we ignore the scaling factor of the denominator in the definition (2.2)):

$$C_{\xi\eta}^{H^1} = \frac{1}{8}(R_{k,l,-k,-l} + R_{-k,l,k,-l})$$

The computation gives an explicit formula

$$C_{\xi\eta}^{H^1} = \frac{S}{36}(k \times l)^2 \left(4A^\alpha(k) + 4A^\alpha(l) - 3A^\alpha(k+l) - 3A^\alpha(k-l) + \frac{(A^\alpha(k) - A^\alpha(l))^2}{A^\alpha(k-l)} + \frac{(A^\alpha(k) - A^\alpha(l))^2}{A^\alpha(k+l)}\right)$$

which we rewrite in the following form

$$C_{\xi\eta}^{H^1} = \rho^2\{A^\alpha(k+l)A^\alpha(k-l)(4A^\alpha(k) + 4A^\alpha(l) - 3A^\alpha(k+l) - 3A^\alpha(k-l)) + (A^\alpha(k) - A^\alpha(l))^2(A^\alpha(k+l) + A^\alpha(k-l))\}, \quad (3.6)$$

where $\rho^2 = \dfrac{S(k \times l)^2}{36A^\alpha(k+l)A^\alpha(k-l)}$ is a function of k, l, α and is strictly positive. Hence, the sign of the curvature is determined by the expression in the bracket, which is a cubic polynomial in α^2:

$$B(\alpha, k, l) \equiv b_0 + b_1\alpha^2 + b_2(\alpha^2)^2 + b_3(\alpha^2)^3, \quad (3.7)$$

so that $C_{\xi\eta}^{H^1} = \rho^2 B(\alpha, k, l)$.

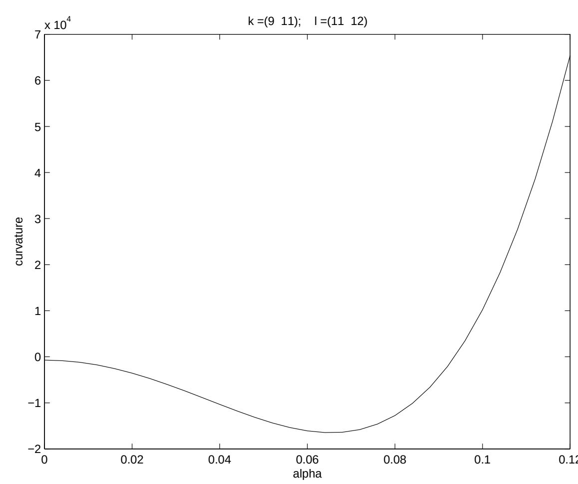

FIGURE 3.1. Sectional curvature (3.6) as a function of α for the case $k = (9, 11), l = (11, 12)$.

Numerical analysis of this complicated expression shows that the sectional curvature becomes positive for some values of $\alpha > \alpha_0$ when $k - l$ is small. Fig. (3.1) is representative of a typical behavior of the curvature as a function of α for $l = k + \epsilon$, where $\epsilon \ll k$ is small. Based on this numerical evidence we analyze further analytically the case $l = k + \epsilon$, where $\epsilon \ll k$ is small. Compute the coefficients b_n in (3.7) as power series in ϵ

$$b_0 = -64k^4\epsilon^2 + 16k^2(k, \epsilon)^2 + \mathcal{O}(\epsilon^4) \quad (3.8)$$

$$b_1 = -224k^6\epsilon^2 + 128k^4(k,\epsilon)^2 + \mathcal{O}(\epsilon^4) \tag{3.9}$$
$$b_2 = -640k^8\epsilon^2 + 320k^6(k,\epsilon)^2 + \mathcal{O}(\epsilon^4) \tag{3.10}$$
$$b_3 = 256k^8(k,\epsilon)^2 + \mathcal{O}(\epsilon^4) \tag{3.11}$$

Notice that the coefficient of the highest degree is positive while all the rest are negative. Hence, for $k > 1/\alpha$ it defines the leading term which increases with α, while the other coefficients are responsible for initial decrease seen in Fig. (3.1). We summarize our result in the following theorem.

Theorem 3.2. *Consider the sectional curvature of the group $\mathcal{D}_\mu(T^2)$ equipped with the right invariant H^1 metric in the plane defined by the stream functions $\xi = \cos(k,x)$ and $\eta = \cos(l,x)$, where $l = k+\epsilon$. Then, for $|\epsilon|$ sufficiently small, for any k there is an $0 < \alpha_0(k) < 1$, such that for all $\alpha > \alpha_0(k)$ the corresponding sectional curvature is positive.*

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